

Analytical approach to derivatives exposure with Mathematica 10

Financial derivatives exposures, as shown in the previous document, can be measured analytically or numerically using Monte Carlo simulation. Whilst numerical approach is much more common, analytical routines have also their own attractions – they are elegant, tractable and tidy. Being analytic, they are also fast and quick to implement. Functional format of the exposure facilitates intuitive calculation of risk, so risk sensitivities can be obtained for any input variable. Analytical computation of exposures is one of the key strengths of **Mathematica 10** where powerful symbolic engine can be applied to obtain the desired result.

Our objective is to obtain the exposure profile of derivative contract – say interest rate swap with 5Y maturity and transacted fixed rate $S = 3.5\%$.

Contract value: $V = (F - S) * Duration$ where F is the future swap rate and S is the transacted swap rate.

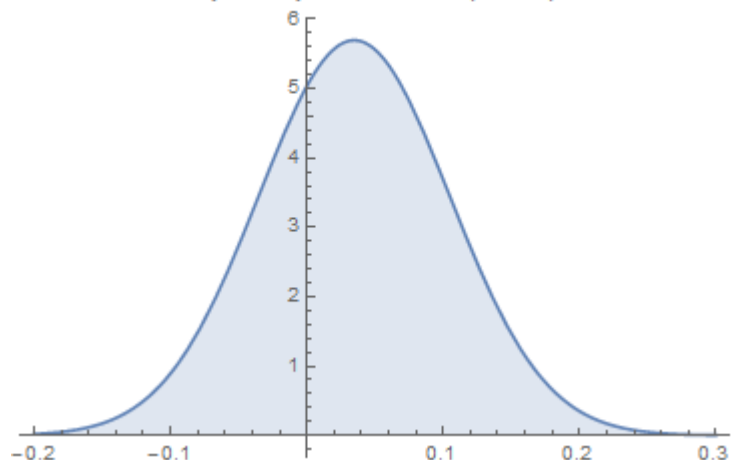
Future swap rate [F]: here we need to specify what SDE process the swap rate F will follow in order to get back the result. We will explore three cases:

1. Mean-reverting normal process (Ornstein-Uhlenbeck process)
2. Standard normal process (Brownian motion)
3. Logarithmic process (Geometric Brownian motion)

```
ouproc = OrnsteinUhlenbeckProcess[μ, σ, θ, x0];
nrproc = ItoProcess[{μ, σ}, {x, x0}, t];
gbproc = GeometricBrownianMotionProcess[μ, σ, x0];
```

We can use the process definition to get the probability density of particular model for the swap rate. For example, for the **Mean-reverting normal process**, we get:

Probability density of the MR Swap Rate process



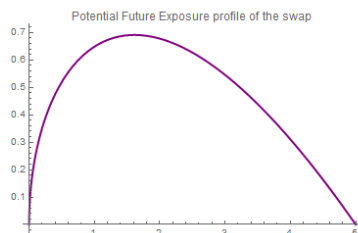
Knowing the distribution of the SDE process gives us an opportunity to define various measures analytically.

For example- **Potential Future Exposure [PFE]** of the 5Y swap:

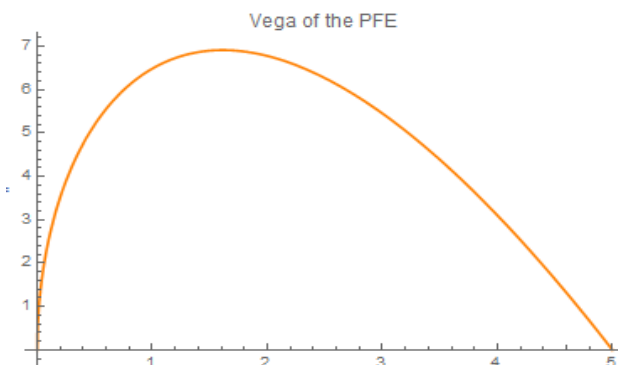
```
itd = Refine[InverseCDF[td, α], 0 ≤ α ≤ 1]
e^{-tθ} (t - T) (-x0 + e^{tθ} (s - μ) + μ) - (t + T) \sqrt{\frac{1 - e^{-2tθ}}{θ}} σ InverseErfc[2 α]
```

And it's profile as function of time at 95% confidence level:

```
Plot[itd /. {α → 0.95, s → 0.035, T → 5, μ → 0.04, σ → 0.1, θ → 0.03, x0 → 0.035}, {t, 0, 5},
PlotLabel → "Potential Future Exposure profile of the swap",
PlotStyle → {Thick, Purple}]
```



As we can see, the PFE increases with time. We can calculate any derivative we want – say 'vega' of the $PFE(t)$ – the sensitivity to the swap rate volatility:



We can define the **exposure function** as follows:

$$\text{exposure} = (T - t) * \text{If}[x[t] > s, x[t] - s, 0];$$

Where $x[t]$ is the future swap rate and s is the transacted fixed rate on the contract.

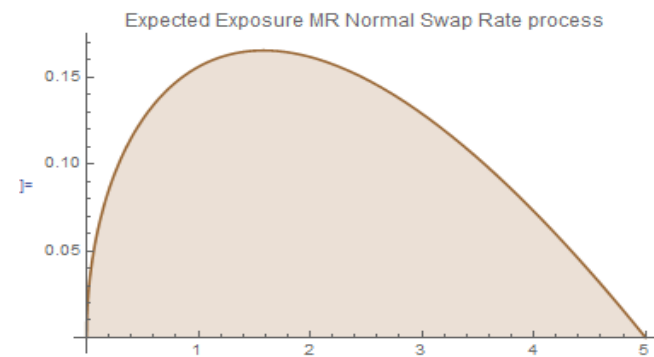
Exposure function is a practical way to calculate the **Expected Exposure [EE]** – or the mean exposure of contract over its life. Since Mathematica 10 'Expectation' function accept SDE process as parameter, this leads to a quick result.

The **EE of the MR normal process:**

```
EE = Expectation[exposure, x ≈ ouproc,
Assumptions → {T > 0, s > 0, σ > 0, t > 0}] // Simplify
```

$$\frac{1}{2\sqrt{\pi}\theta} \sqrt{\frac{e^{2t\theta}-1}{\theta}} \frac{\theta}{\sqrt{e^{2t\theta}-1}} (t-T) \exp\left(-\frac{\theta(\mu^2 + e^{2t\theta}(\mu^2 + 2\sigma^2 t) + 2\mu(s+x0)e^{t\theta} - 2\sigma^2 t + x0^2)}{\sigma^2(e^{2t\theta}-1)}\right) \sqrt{\frac{\theta}{2\pi(e^{2t\theta}-1)}} \left(\frac{\mu + (s-\mu)e^{t\theta} - x0}{\sigma}\right) + (s0 - \mu) \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(\mu^2 + 2\sigma^2 t) + 2\mu(s+x0)e^{t\theta} - \sigma^2 t + x0^2)}{\sigma^2(e^{2t\theta}-1)}\right) + (\mu - s) \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(\mu^2 + 2\sigma^2 t) + 2\mu(s+x0)e^{t\theta} - 2\sigma^2 t + x0^2)}{\sigma^2(e^{2t\theta}-1)}\right) + \sqrt{\pi}\theta(s-\mu) \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(\mu^2 + 2\sigma^2 t) + 2\mu(s+x0)e^{t\theta} - 2\sigma^2 t + x0^2)}{\sigma^2(e^{2t\theta}-1)}\right) - \sqrt{\pi}\theta(s0 - \mu) \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(\mu^2 + 2\sigma^2 t) + 2\mu(s+x0)e^{t\theta} - \sigma^2 t + x0^2)}{\sigma^2(e^{2t\theta}-1)}\right) + \sigma \sqrt{\frac{\theta}{e^{2t\theta}-1}} \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(2\mu s + 3\sigma^2 t) + 2e^{t\theta}(\mu^2 + s x0) + \sigma^2(-t) + 2\mu x0)}{\sigma^2(e^{2t\theta}-1)}\right) - \sigma \sqrt{\frac{\theta}{e^{2t\theta}-1}} \exp\left(\frac{\theta(\mu^2 + e^{2t\theta}(2\mu s + 3\sigma^2 t) + 2e^{t\theta}(\mu^2 + s x0) - 3\sigma^2 t + 2\mu x0)}{\sigma^2(e^{2t\theta}-1)}\right)$$

And the exposure profile displayed graphically:



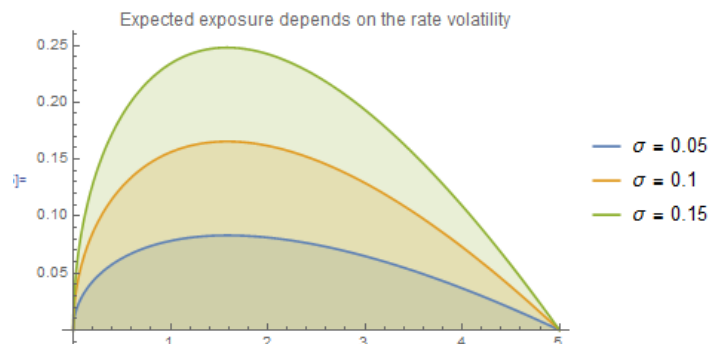
The analytical EE is very similar to what can be obtained from MC simulation. However, a big advantage of analytical method is immediate availability of 'what-if' scenarios.

For example, we can visualise the EE profile for various volatility inputs:

```

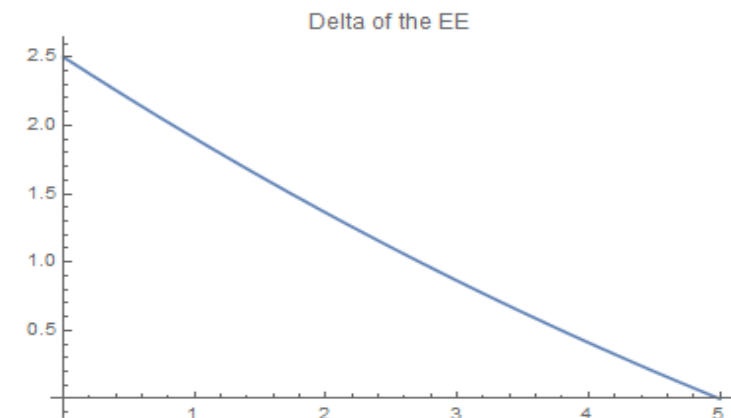
vols = {0.05, 0.1, 0.15};
eenum2 = EE /. {T -> 5, s -> 0.035, mu -> 0.04, theta -> 0.05, x0 -> 0.035};
Plot[Evaluate@Table[eenum2, {sigma, vols}], {t, 0, 5}, Filling -> Axis,
PlotLegends -> {StringJoin["sigma = ", ToString[#]] & /@ vols},
PlotLabel -> "Expected exposure depends on the rate volatility"]

```



As the graph above highlights, the expected exposure is very sensitive to the volatility of the swap rate. The higher the volatility, the higher the EE.

Other sensitivities can be obtained in a similar way. This is, for example, the 'delta' of the EE as the function of time:



Similar to the MC case, we can calculate the **Expected positive exposure [EPE]** – which is the time-weighted EE over the life of the contract:

$$EPE(t) = \frac{1}{T} \int EE(t) dt$$

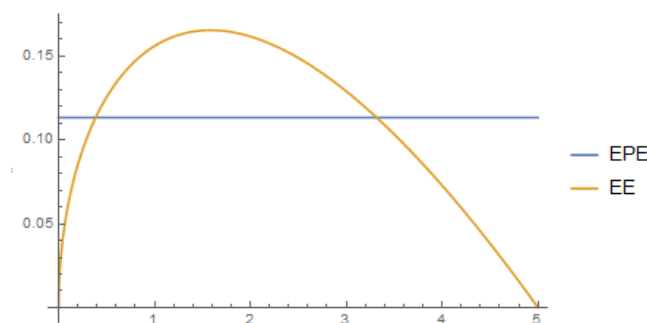
Since the Expected exposure function for the MR process is quite complex, symbolic integration will be time-consuming. We therefore resort to the numerical integration. For the 5Y contract with 3.5% fixed rate, 10% volatility and 5% MR speed the EPE is 11.34% of the notional value:

```

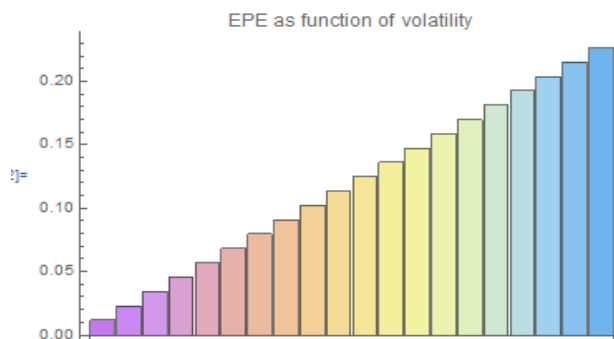
EPE = NIntegrate[1/T EE /. {T -> 5, s -> 0.035, mu -> 0.04, sigma -> 0.1, theta -> 0.05, x0 -> 0.035}, {t, 0, 5}]
0.113412

```

And graphically:



Similar to other measures, EPE is sensitive to its input parameters. For example, the key driver again is the swap rate volatility:



Effective expected exposure [EEE] – this is the maximum EE over the time horizon:

$$EEE(t) = \text{Max}[EE(t)] \quad 0 \leq s \leq t$$

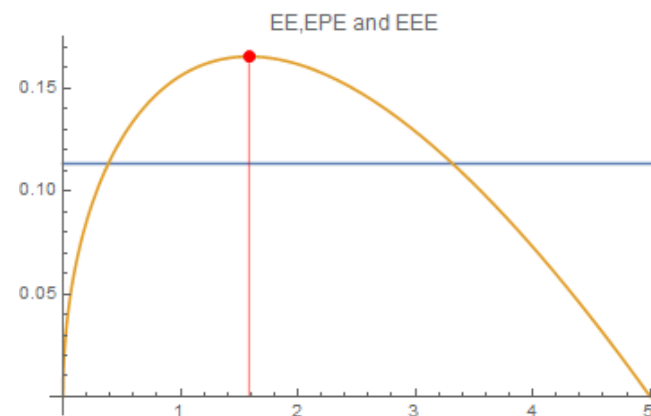
We can easily get it by using FindMaximum function:

```

eenum = EE /. {T -> 5, s -> 0.035, mu -> 0.04, sigma -> 0.1, theta -> 0.05, x0 -> 0.035};
max = FindMaximum[eenum, {t, 0.1}]
{0.165594, {t -> 1.58336}}

```

The EEE = 16.6% and occurs at t=1.6 years from now:



Although we worked with the Ornstein-Uhlenbeck mean-reverting process, the same approach will work with other model specifications:

The Expected Exposure for other SDE models:

- 1) Standard **BM normal model**:

$$-\frac{1}{2}(t-T)e^{-\frac{(-s+\mu t+x_0)^2}{2\sigma^2 t}} \left(-(s-\mu t-x_0)e^{-\frac{(-s+\mu t+x_0)^2}{2\sigma^2 t}} \operatorname{erf}\left(\frac{-s+\mu t+x_0}{\sqrt{2}\sigma\sqrt{t}}\right) + (-s+\mu t+x_0)e^{-\frac{(-s+\mu t+x_0)^2}{2\sigma^2 t}} + \sqrt{\frac{2}{\pi}}\sigma\sqrt{t} \right)$$

- 2) **Lognormal GBM model**:

$$\frac{1}{2}(t-T) \left(s \operatorname{erfc}\left(\frac{2\log(s)-2\mu t+\sigma^2 t-2\log(x_0)}{2\sqrt{2}\sigma\sqrt{t}}\right) + x_0 e^{\mu t} \left(\operatorname{erfc}\left(\frac{-2\log(s)+2\mu t+\sigma^2 t+2\log(x_0)}{2\sqrt{2}\sigma\sqrt{t}}\right) - 2 \right) \right)$$